Stochastic Properties of the Random Waypoint Mobility Model

Christian Bettstetter* (bettstetter@ei.tum.de)
Technische Universität München, Institute of Communication Networks, 80290 Munich, Germany

Hannes Hartenstein, Xavier Pérez–Costa
NEC Europe, Network Laboratories, 69115 Heidelberg, Germany

Abstract. The random waypoint model is a commonly used mobility model for simulations of wireless communication networks. By giving a formal description of this model in terms of a discrete–time stochastic process, we investigate some of its fundamental stochastic properties with respect to: (a) the transition length and time of a mobile node between two waypoints, (b) the spatial distribution of nodes, (c) the direction angle at the beginning of a movement transition, and (d) the cell change rate if the model is used in a cellular–structured system area.

The results of this paper are of practical value for performance analysis of mobile networks and give a deeper understanding of the behavior of this mobility model. Such understanding is necessary to avoid misinterpretation of simulation results. The movement duration and the cell change rate enable us to make a statement about the 'degree of mobility' of a certain simulation scenario. Knowledge of the spatial node distribution is essential for all investigations in which the relative location of the mobile nodes is important. Finally, the direction distribution explains in an analytical manner the effect that nodes tend to move back to the middle of the system area.

Keywords: mobility modeling, modeling and simulation, analysis of mobile networks, random waypoint model

Accepted for *ACM/Kluwer Wireless Networks*, Special Issue on Modeling & Analysis of Mobile Networks.
Authors' preprint. Mar 10, 2003.

1. Introduction and Motivation

Mobility models are important building blocks in simulation–based studies of wireless networks. Researchers in this area can choose from a variety of models that have been developed in the wireless communications and mobile computing community during the last decades [4, 13, 18, 22, 29]. Moreover, well–known motion models from physics and chemistry — such as random walk or Brownian motion — and models from transportation theory [8, 15, 16] are used in simulations of mobile networks. Surveys and classifications on this topic can be found in [4, 16, 19, 20, 29].

A very popular and commonly used mobility model is the *random waypoint (RWP) model*. It is implemented in the network simulation tools ns–2 [2] and GloMoSim [1] and used in several performance evaluations of ad hoc networking protocols [9, 11, 17]. This mobility model is a simple and straightforward stochastic model that describes the movement behavior of a

* Corresponding author. Ph: +49 89 289 25813, Fax: 23523

mobile network node in a given system area as follows (see Fig. 1): A node randomly chooses a destination point ('waypoint') in the area and moves with constant speed on a straight line to this point. After waiting a certain pause time, it chooses a new destination and speed, moves with constant speed to this destination, and so on. The movement of a node from a starting position to its next destination is denoted as one *movement period* or *transition* in this paper. The destination points are uniformly randomly distributed on the system area.

In previous work [4], Bettstetter noted that a fundamental knowledge of the behavior of this model is important to interpret simulation results correctly. Although many researchers use the RWP model in their simulations, a basic understanding of its impact on the simulation results is still lacking. The paper at hand addresses this issue. It presents a detailed investigation of some stochastic characteristics of the RWP model.

Our contributions are as follows: In Section 2 we define the RWP model in a formal manner as a discrete–time stochastic process. Based on this description, Section 3 derives typical stochastic parameters of a node's traveled distance and time during one movement transition by making use of some classical results of geometrical probability theory. In particular, we give equations for the expected value, variance, and probability density function of the transition length in a circular and rectangular system area. Next, we discuss the mapping from transition length to duration, where we consider scenarios with and without pause time at the destination waypoints. We are interested in this duration because it defines a 'mobility metric' that can be used to describe a certain simulation scenario. Section 4 discusses in detail the spatial distribution of nodes moving according to the RWP model. It was already mentioned in previous papers that this distribution is non–uniform (see, e.g., [4, 25]). This property has important consequences for all simulation–based studies in which the relative position of nodes is of relevance (e.g., studies of power control, medium access control protocols). Thus, knowing the spatial node distribution of the RWP model is an essential requirement for investigations in this field. In Section 5, we calculate the probability density function of the movement direction of a node at a given location. This direction distribution describes analytically the effect reported in [4, 25] that RWP nodes tend to move back from the border to the middle of the area. In Section 6, we employ the RWP model to study the movement of mobile stations in a cellular–structured network. Using a simple, rectangular cell layout, we investigate the expected number of cell changes per movement period and the expected number of cell changes per unit time, i.e., the cell change rate. Section 7 outlines related work on the random waypoint model, and, finally, Section 8 summarizes our main results.

2. Definition of the RWP Stochastic Process

This paper studies random waypoint movement in a one– or two–dimensional system space $\mathcal{A}$. In one dimension we consider a line segment; in two dimensions we consider a rectangular area of size $a \times b$ or a circular area with radius a. For proper nomenclature, several random variables must be defined. These variables are written in upper case letters, whereas specific outcomes are written in lower case. Multi–dimensional variables (e.g., random coordinates in an area) are written in bold face; scalar variables (e.g., random lengths) are in normal font. The parameter j identifies a particular node, and the discrete time parameter i denotes the movement period of this node.

The random variable representing the Cartesian coordinates of the waypoint that a node j chooses in its movement period i is denoted by the vector $\mathbf{P}_i^{(j)}$. With this definition, the movement trace of an RWP node j can be formally described as a discrete–time stochastic process, given by selecting a random waypoint $\mathbf{P}_i^{(j)}$ for each movement period i:

$$\left\{\mathbf{P}_i^{(j)}\right\}_{i\in\mathbb{N}_0} = \mathbf{P}_0^{(j)}, \mathbf{P}_1^{(j)}, \mathbf{P}_2^{(j)}, \mathbf{P}_3^{(j)}, \ldots . \tag{1}$$

These waypoints are independently and identically distributed (i.i.d.) using a uniform random distribution over the system space $\mathcal{A}$. Since each node moves independently of other nodes, it is sufficient to study the movement process of a single node. Thus, we often omit the index j.

Let us now consider the case that a node randomly chooses a new speed V_i for movement from $\mathbf{P}_{i-1}$ to $\mathbf{P}_i$ and a pause time $T_{p,i}$ at waypoint $\mathbf{P}_i$. The complete movement process of a node is then given by

$$\{(\mathbf{P}_i, V_i, T_{p,i})\}_{i\in\mathbb{N}} = (\mathbf{P}_1, V_1, T_{p,1}), (\mathbf{P}_2, V_2, T_{p,2}), (\mathbf{P}_3, V_3, T_{p,3}), \ldots , \tag{2}$$

where an additional waypoint $\mathbf{P}_0$ is needed for initialization. A sample of this process is denoted by $\{(\mathbf{p}_i, v_i, \tau_{p,i})\}_{i\in\mathbb{N}}$. A movement period i can be completely described by the vector $(\mathbf{p}_{i-1}, \mathbf{p}_i, v_i, \tau_{p,i})$. When we just refer to a single random variable of a process, we omit the index i and just write $\mathbf{P}$, V, or T_p. The values for the pause time are chosen from a bounded random distribution $f_{T_p}(\tau_p)$ in the interval $[0, \tau_{p,max}]$ with $\tau_{p,max} < \infty$ and a well–defined expected value $E\{T_p\}$. In general the speed is also chosen from a random distribution $f_V(v)$ within the interval $[v_{min}, v_{max}]$ with $v_{min} > 0$ and $v_{max} < \infty$.

3. Transition Length and Duration

In simulation–based research on wireless networks it is often desired to compare simulation results that have been obtained using different random mobility models. To do so, one should define a metric for the 'degree of mobility' of the simulated scenario. Due to the broad range of mobility models used in the literature and their various parameters, such a definition is not trivial. Nevertheless, we can state that two mobility parameters are of major interest in all models:

- Speed behavior: Is there a constant speed or a speed distribution? When and how does a node change its speed?
- Direction change behavior: What is the frequency of direction changes? How does a node change its direction?

For example, in the commonly used *random direction model* as described in [4], the time between two direction change events is taken from an exponential distribution. This time is independent of the speed of the node; and, if a wrap–around border behavior [3] is used, it is also independent of the size and shape of the area.

As opposed to this, the RWP model correlates the speed and the direction change behavior. The time between two direction change events is no longer an adjustable input parameter of the model, but it depends on the speed of the nodes and the size and shape of the area. For a given area, a higher speed results in a higher frequency of direction changes.

In this section, we thus investigate the time between two direction change events of the RWP model. We first regard the transition length, i.e., the Euclidian distance that a node travels during one movement period between two waypoints. We define the sequence of these distances as a stochastic process and show its ergodic properties. Afterward, we give analytical expressions of its probability density function (pdf) and the first two moments. Hereby, we first investigate an RWP model in one dimension and then consider rectangular and circular areas. Finally, we explain the conversion from transition length to time and discuss the impact of the pause time. Among other results, we obtain equations for the average duration of an RWP period with uniform and discrete speed distribution of the nodes.

3.1. Stochastic Process of Transition Lengths

As defined above, the stochastic process representing the RWP movement of a node j is given by the sequence of random waypoints $\mathbf{P}_0^{(j)}, \mathbf{P}_1^{(j)}, \ldots$. The corresponding stochastic process of distances between two consecutive waypoints is given by

$$\left\{L_i^{(j)}\right\}_{i \in \mathbb{N}} = L_1^{(j)}, L_2^{(j)}, L_3^{(j)}, \ldots \quad \text{with} \quad L_i^{(j)} = \left\| \mathbf{P}_i^{(j)} - \mathbf{P}_{i-1}^{(j)} \right\| . \tag{3}$$

A sample of this process is written as $\{l_i^{(j)}\}$. While the random waypoints are i.i.d. per definition, the distances are not stochastically independent, essentially because the endpoint of one movement period i is the starting point of the successive movement period $i+1$.

We are now interested in the expected value of L. It can be interpreted in two ways:

$$E\{L\} = \underbrace{\lim_{m \to \infty} \frac{1}{m} \sum_{i=1}^{m} l_i^{(j)}}_{\text{time average of node } j} = \underbrace{\lim_{n \to \infty} \frac{1}{n} \sum_{j=1}^{n} l_i^{(j)}}_{\text{ensemble average at period } i} . \tag{4}$$

In words, the *time average* of the transition lengths experienced by a single RWP node j over a long–run simulation ($m \to \infty$) is equal to the *ensemble average* of one period i in an RWP process with many nodes ($n \to \infty$). In the nomenclature of random processes, we thus have a mean–ergodic property of the RWP mobility model.

While this result is intuitively convincing, the proof is not trivial since the random variables $L_1, L_2, L_3, \ldots$ are not stochastically independent. But despite the fact that 'last endpoint equals next starting point,' the ergodic property holds as can be seen as follows. Assume we look at a random process given by

$$\{L_{2i-1}\}_{i\in\mathbb{N}} \;=\; L_1,\, L_3,\, L_5\, \ldots, \tag{5}$$

i.e., we look only at every second variable of the original process of random distances. The length L_1 is a deterministic function of $\mathbf{P}_0$ and $\mathbf{P}_1$, the length L_3 a deterministic function of $\mathbf{P}_2$ and $\mathbf{P}_3$, and so on. Since $\mathbf{P}_0, \mathbf{P}_1, \mathbf{P}_2, \ldots$ are i.i.d. random variables, it follows that $L_1, L_3, \ldots$ are also independent. The mean–ergodic property is obtained immediately in this sub–process because both the time and ensemble average is formed by mutually independent variables. The same is true for the sub–process

$$\{L_{2i}\}_{i\in\mathbb{N}} \;=\; L_2,\, L_4,\, L_6\, \ldots\,. \tag{6}$$

Now, combining these two sub–processes does not change the asymptotic behavior of the time averages of the combined process, thus, Equation (4) holds. In the same manner one also obtains the distribution–ergodic property of the process.

With respect to our problem the above mentioned ergodic properties imply the following: in order to determine the distribution or expected value of the transition length and time observed by a single node following the RWP model, the analysis can be simplified by considering only the distribution of the distance between *two independent points* placed uniformly at random in the system area. This observation allows the transfer of results of geometrical probability theory to our problem. In the following, we make no difference in notation between the 'distance between two consecutive waypoints' and the 'distance between two independent random points sampled from a uniform distribution.' Both are represented by the random variable L.

3.2. Transition Length on One–Dimensional Line Segment

We first consider a one–dimensional line segment $[0, a]$. Two random points are uniformly placed on this segment, i.e., the pdf of a point's location $\mathbf{P} = P_x$ is

$$f_{P_x}(x) \;=\; \begin{cases} 1/a & \text{for } 0 \le x \le a \\ 0 & \text{else} \end{cases}\,. \tag{7}$$

Since both points are independent from each other, their joint pdf is

$$f_{P_{x1}P_{x2}}(x_1, x_2) \;=\; f_{P_x}(x_1)\, f_{P_x}(x_2) \;=\; \begin{cases} 1/a^2 & \text{for } 0 \le x_1, x_2 \le a \\ 0 & \text{else} \end{cases}\,. \tag{8}$$

The distance between two random points is defined by $L = |P_{x1} - P_{x2}|$. The probability that this distance is smaller than a given value l can be computed by the integral of the joint pdf over the area defined by $\mathcal{D} = |x_1 - x_2| \le l$ in the x_1–x_2–space, i.e.,

$$P\,(L \le l) \;=\; \iint\limits_{\mathcal{D}} f_{P_{x1}P_{x2}}(x_1, x_2)\, dx_2\, dx_1\;\;, \tag{9}$$

for $0 \le l \le a$. Clearly, $P(L \le l) = 1$ for $l > a$. Taking into account the bounds of both $\mathcal{D}$ and $f_{P_{x1}P_{x2}}(x_1, x_2)$, we obtain the cumulative distribution function (cdf)

$$P(L \le l) \;=\; \frac{1}{a^2}\left(\int_0^l \int_0^{x_1+l} dx_2\, dx_1 + \int_l^{a-l} \int_{x_1-l}^{x_1+l} dx_2\, dx_1 + \right.$$
$$\left. + \int_{a-l}^{a} \int_{x_1-l}^{a} dx_2\, dx_1 \right) \;=\; -\frac{1}{a^2}\, l^2 + \frac{2}{a}\, l\,. \tag{10}$$

The derivative of this function with respect to l yields by definition the desired pdf

$$f_L(l) \;=\; \frac{\partial}{\partial\, l}\, P(L \le l) \;=\; -\frac{2}{a^2}\, l + \frac{2}{a} \tag{11}$$

for $0 \le l \le a$, and $f_L(l){=}0$ otherwise. The expected distance is

$$E\{L\} = \int_0^a l\, f_L(l)\, dl \;=\; \frac{1}{3}\, a, \tag{12}$$

and its variance yields

$$E\left\{L^2\right\} = \int_0^a l^2\, f_L(l)\, dl = \frac{1}{6}\, a^2. \tag{13}$$

With the above results on ergodicity, these stochastic properties of the distances between a pair of independently uniformly distributed points also represent the stochastic properties of the moved distance of an RWP node within one period.

3.3. Transition Length in Rectangular Area

Let us now consider RWP movement in a rectangular area of size $a \times b$ and again derive the distribution of the transition length L. Without loss of generality we assume $a \ge b$. The spatial distribution of the two–dimensional waypoints $\mathbf{P} = (P_x, P_y)$ is now given by the uniform distribution

$$f_{P_xP_y}(x, y) = \begin{cases} 1/(ab) & \text{for } 0 \le x \le a \text{ and } 0 \le y \le b \\ 0 & \text{else} \end{cases}\,. \tag{14}$$

The distance between two points $\mathbf{P}_1 = (P_{x1}, P_{y1})$ and $\mathbf{P}_2 = (P_{x2}, P_{y2})$ is

$$L = \|\, \mathbf{P}_2 - \mathbf{P}_1\| = \sqrt{|P_{x1} - P_{x2}|^2 + |P_{y_1} - P_{y_2}|^2} \;=\; \sqrt{L_x^2 + L_y^2}\,. \tag{15}$$

Note that the random variable $L_x = |P_{x1} - P_{x2}|$ represents the random distance between two uniformly distributed coordinates P_{x1} and P_{x2} on a one–dimensional line segment $[0, a]$. Thus, its pdf is given by Equation (11). The same holds for $L_y = |P_{y_1} - P_{y_2}|$ if we replace a by b. In addition, both random distances are independent of each other, and therefore the joint pdf of L_x and L_y is given by

$$f_{L_xL_y}(l_x, l_y) \;=\; f_L(l_x)\, f_L(l_y) \;=\; \frac{4}{a^2b^2}\,(\,-\,l_x + a)(\,-\,l_y + b) \tag{16}$$

for $0 \le l_x \le a$ and $0 \le l_y \le b$, and 0 otherwise. Knowing this expression, we can derive the cdf $P(L \le l)$ by integration of $f_{L_xL_y}(l_x, l_y)$ over the circle area $\mathcal{D} = l_x^2 + l_y^2 \le l$ in the l_x–l_y–space, i.e,

$$P\,(L \le l) \;=\; \iint_{\mathcal{D}} f_{L_xL_y}(l_x, l_y)\, dl_y\, dl_x \;\;. \tag{17}$$

As in the one–dimensional case, we cannot compute this integral in a straightforward manner, namely by setting the right hand side of (16) in (17), but must take into account that $f_{L_xL_y}(l_x, l_y) = 0$ for $l_x > a$ or $l_y > b$. Thus, we distinguish between three cases:

$$P\,(L \le l) = \begin{cases} \int_0^l \int_0^{\sqrt{l^2-l_x^2}} f(l_x, l_y)\, dl_y\, dl_x & \text{for } 0 \le l \le b \\ \int_0^b \int_0^{\sqrt{l^2-b^2}} f(l_x, l_y)\, dl_y\, dl_x & \\ \quad + \int_0^{\sqrt{l^2-l_x^2}} \int_{\sqrt{l^2-b^2}}^{l} f(l_x, l_y)\, dl_y\, dl_x & \text{for } b < l < a \\ \int_0^b \int_0^{\sqrt{l^2-b^2}} f(l_x, l_y)\, dl_y\, dl_x & \\ \quad + \int_0^{\sqrt{l^2-l_x^2}} \int_{\sqrt{l^2-b^2}}^{a} f(l_x, l_y)\, dl_y\, dl_x & \text{for } a \le l \le \sqrt{a^2+b^2} \end{cases} \tag{18}$$

Solving these integrals, taking the derivate with respect to l, and performing some trigonometric simplifications, leads to the following result.

Result. The pdf of the transition length L of nodes moving according to the RWP model in a rectangular area of size $a \times b$, $a \ge b$, is

$$f_L(l) = \frac{4l}{a^2b^2} \cdot f_0(l) \tag{19}$$

with

$$f_0(l) = \begin{cases} \frac{\pi}{2}\, ab - al - bl + \frac{1}{2} l^2 & \text{for } 0 \le l \le b \\ ab\, \arcsin \frac{b}{l} + a\sqrt{l^2 - b^2} - \frac{1}{2}\, b^2 - al & \text{for } b < l < a \\ ab\, \arcsin \frac{b}{l} + a\sqrt{l^2 - b^2} - \frac{1}{2}\, b^2 - & \\ \quad ab\, \arccos \frac{a}{l} + b\sqrt{l^2 - a^2} - \frac{1}{2}\, a^2 - \frac{1}{2}\, l^2 & \text{for } a \le l \le \sqrt{a^2+b^2} \\ 0 & \text{otherwise} \end{cases} \;. \tag{20}$$

The same result of the distance pdf between two random points was derived by Ghosh in 1951 [12] using a transformation to polar coordinates.

Introducing the normalized random length $\hat{L} = L/a$, Figure 3.3 shows $f_{\hat{L}}(\hat{l})$ for areas of size $a \times a$ and $a \times \frac{a}{2}$. For arbitrary a, the value of $f_L(l)$ is obtained by $f_L(l) = \frac{1}{a} f_{\hat{L}}(\hat{l})$. The expected value of L is [12]

$$\begin{aligned} E\,\{L\} = {} & \frac{1}{15}\left[\frac{a^3}{b^2} + \frac{b^3}{a^2} + \sqrt{a^2+b^2}\left(3 - \frac{a^2}{b^2} - \frac{b^2}{a^2}\right)\right] \\ & + \frac{1}{6}\left[\frac{b^2}{a}\, \text{arcosh} \frac{\sqrt{a^2+b^2}}{b} + \frac{a^2}{b}\, \text{arcosh} \frac{\sqrt{a^2+b^2}}{a}\right] \end{aligned} \tag{21}$$

with $\mathrm{arcosh}(x) = \ln\left(x + \sqrt{x^2 - 1}\right)$. Figure 3.3 shows the curve for $E\{L\}/a$ over b/a. For example, the expected length within a square of size $a \times a$ is $E\{L\} = 0.5214\,a$, and a rectangle of size of $a \times \frac{a}{2}$ yields $E\{L\} = 0.402\,a$. The variance of L is given by

$$E\left\{L^2\right\} = \frac{1}{6}\left(a^2 + b^2\right) . \tag{22}$$

Note that for $b \to 0$ the moments for the one–dimensional case are obtained, i.e., $\lim_{b\to 0} E\{L\} = \frac{1}{3}\,a$ and $\lim_{b\to 0} E\{L^2\} = \frac{1}{6}\,a^2$.

3.4. Transition Length in Circular Area

On a circular system area of radius a, the distance pdf $f_L(l)$ can be derived, for example, as follows. We regard a node with a given starting waypoint $\mathbf{P} = \mathbf{p}$ and compute the conditional probability $P(L \le l \mid \mathbf{P} = \mathbf{p})$ using basic geometric equations on the intersection area of two circles. Hereby, we use polar coordinates. Straightforward integration over all possible starting waypoints in the area gives $P(L \le l)$, whose derivate is $f_L(l)$. These operations lead to the following result, which is also reported in [14].

Result. The pdf of the transition length L of nodes moving according to the RWP model on a disk of radius a is given by

$$f_L(l) = \frac{8}{\pi a}\frac{l}{2a}\left(\arccos\frac{l}{2a} - \frac{l}{2a}\sqrt{1 - \left(\frac{l}{2a}\right)^2}\right) , \tag{23}$$

for $0 \le l \le 2a$ and 0 otherwise.

Again, we can introduce a normalized random variable $\hat{L} = L/2a$ for simplicity. Figure 3.4 shows the plot of $f_{\hat{L}}(\hat{l})$. The expected value of L is

$$E\{L\} = \int_0^{2a} l\, f_L(l)\, dl = \frac{128}{45\,\pi} \cdot a \;=\; 0.9054\,a\ , \tag{24}$$

and its variance is

$$E\left\{L^2\right\} = \int_0^{2a} l^2\, f_L(l)\, dl = a^2. \tag{25}$$

3.5. Transition Time

Let us now employ our results on the transition length to calculate the stochastic properties of the transition time, i.e., the time it takes a node to move from one waypoint to the next waypoint. The corresponding random variable is denoted by T and an outcome is written as τ.

If the speed of a node is constant during the entire movement process, i.e., $V_i = v = const\,\forall i$ and $v > 0$, we have

$$T \;=\; \frac{1}{v}\,L\,. \tag{26}$$

Hence, the expected transition time is

$$E\{T\} = \frac{1}{v}\, E\{L\} \;, \tag{27}$$

and its pdf can be computed by

$$f_T(\tau) = v\, f_L(v\tau) \;, \tag{28}$$

with $E\{L\}$ and f_L taken from (20–21) or (23–24), respectively.

We now consider the case in which the speed of a node is not constant but chosen from a random distribution $f_V(v)$ at each waypoint (and then stays constant during one transition). We require $v_{min} \le V \le v_{max}$ and $v_{min} > 0$ and can write

$$T = \frac{L}{V} \;. \tag{29}$$

In this case, the random variable T is formed as a function $g(L,V) = L/V$ of two random variables L and V. In general, the expected value of a variable $g(L,V)$ can be expressed in terms of the joint pdf $f_{LV}(l,v)$ as [23]

$$E\{g(L,V)\} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(l,v)\; f_{LV}(l,v)\, dl\, dv \;. \tag{30}$$

In our case, L and V are independent, and thus their joint pdf is $f_{LV}(l,v) = f_L(l)\, f_V(v)$. The expected value can then be simplified to

$$E\{T\} = E\{L\} \int_{v_{min}}^{v_{max}} \frac{1}{v}\, f_V(v)\, dv \;. \tag{31}$$

The pdf of $T = L/V$ can be computed by

$$f_T(\tau) = \int_{v_{min}}^{v_{max}} v\; f_L(v\tau)\; f_V(v)\; dv \tag{32}$$

for $0 \le \tau \le \tau_{max}$ with $\tau_{max} = l_{max}/v_{min}$, and $f_T(\tau) = 0$ otherwise. Let us explain these results in more detail by using three typical speed distributions: a continuous uniform distribution, a discrete distribution, and a continuous beta distribution.

3.5.1. *Uniform Speed Distribution*

If we employ a uniform speed distribution within $[v_{min}, v_{max}]$, the expected transition time is

$$E\{T\} = \frac{\ln(v_{max}) - \ln(v_{min})}{v_{max} - v_{min}}\, E\{L\} \;. \tag{33}$$

Note that $\lim_{v_{max}\to v_{min}} E\{T\} = E\{L\}/v_{min}$ corresponds to the result (27) for constant speed $v = v_{min} = const$. Further note that the expected time for $v_{min} = 0$ is undefined. This is very reasonable because if a node chooses $V = 0$ the movement transition will take an infinite time. If the maximum speed can be expressed as a multiple of the minimum speed, i.e., $v_{max} = k \cdot v_{min}$, with $k > 1$, we obtain

$$E\{T\} \;=\; \frac{\ln k}{k-1}\,\frac{E\{L\}}{v_{min}}\,. \tag{34}$$

Example. A mobile node moves according to the RWP model on a disk of radius a. It randomly chooses a new speed in each waypoint from a uniform distribution between $[\frac{v_0}{2}, v_0]$. The expected transition time is therefore given by $E\{T\} = 1.2552\,a/v_0$. The pdf can be computed by $f_T(\tau) = \int_{v_0/2}^{v_0} v\, f_L(v\tau)\, \frac{2}{v_0}\, dv$ for $0 < \tau \leq 4a/v_0$, with $f_L(v\tau)$ taken from (23). The resulting function for $a = 1\,\mathrm{m}$ and $v_0 = 1\,\mathrm{m/s}$ is illustrated in Fig. 3.5.1 and compared to a node that moves with deterministic speed $V = v_0$ for all periods. Clearly, the latter node has on average a much shorter transition time, namely $E\{T\} = 0.9054\,a/v_0$, since it always moves with the highest speed v_0. By proper scaling of both axes, we obtain the values for arbitrary a and v_0.

3.5.2. *Discrete Speed Distribution*

We now regard a node that chooses its speed from a set of J discrete values $\{v_1, \ldots, v_j, \ldots, v_J\}$, each value with a certain probability p_j. Clearly, $\sum_{j=1}^{J} p_j = 1$ must hold. Such a discrete speed distribution can be expressed by $f_V(v) = \sum_{j=1}^{J} p_j\, \delta(v - v_j)$, where the function $\delta(x)$ denotes the Dirac delta function. The latter has the properties $\delta(x) = 0$ for $x \neq 0$ and $\delta(0) = \infty$ and $\int_{-\infty}^{\infty} \delta(x) dx = 1$. The expected transition time is then given by

$$E\{T\} \;=\; E\{L\} \sum_{j=1}^{J} \frac{p_j}{v_j}\,. \tag{35}$$

Example. An RWP node moving on a disk chooses in each waypoint either $V = v_1 = v_0/2$ or $v_2 = v_0$, both speed values with equal probability. We thus have $f_V(v) = \frac{1}{2}\left(\delta(v - v_0/2) + \delta(v - v_0)\right)$. The expected transition time of this node is $E\{T\} = 1.3581\,a/v_0$. The pdf of T can be computed with (32) and is shown in Fig. 3.5.1. If the speed value $v_0/2$ is chosen with a probability $p_1 = 0.9$ the average transition time is increased to $E\{T\} = 1.7203\,a/v_0$.

3.5.3. *Beta Speed Distribution*

A non–uniform, continuous speed distribution that is bounded by v_{min} and v_{max} can be expressed in terms of the beta distribution

$$f_V(v) \;=\; \frac{1}{B(\mu_1, \mu_2)} \cdot \frac{v_\star^{\mu_1 - 1}\,(1 - v_\star)^{\mu_2 - 1}}{v_{max} - v_{min}} \tag{36}$$

with $v_\star = \frac{v - v_{min}}{v_{max} - v_{min}}$. The beta function $B(\mu_1, \mu_2)$ is defined by $\int_0^1 z^{\mu_1 - 1}\,(1 - z)^{\mu_2 - 1}\, dz$. Depending on the non–zero parameters μ_1 and μ_2, the function $f_V(v)$ takes a variety of shapes.

For example, if $\mu_1 > 1$ and $\mu_2 > 1$, it has a concave down shape with $f_V(v) \to 0$ for both $V = v_{min}$ and v_{max}. If $\mu_1 = \mu_2$, the curve is symmetric around $\frac{1}{2}(v_{max} - v_{min})$, otherwise the maximum or minimum is shifted closer to v_{max} or v_{min}. For $\mu_1 = \mu_2 = 1$, a uniform distribution is obtained. The average speed is always given by $E\{V\} = \frac{1}{\mu_1+\mu_2}(\mu_1 v_{max} + \mu_2 v_{min})$. Example plots of the beta distribution can be found, for instance, in [23, 28]. Using this large class of speed distributions, the integral in (31) is again solvable, and we can calculate $E\{T\}$ for a given system area in a straightforward manner.

Example. An RWP node in a square area of side a chooses its speed from a beta distribution with parameters $\mu_1 = 4$ and $\mu_2 = 1$ in an interval between $v_{min} = 1\,\text{m/s}$ and $v_{max} = 10\,\text{m/s}$. The expected transition time is $E\{T\} = 0.0665\,a\,\text{s/m}$. For $\mu_1 = 2$ and $\mu_2 = 6$ we obtain $E\{T\} = 0.1892\,a\,\text{s/m}$.

3.6. Time Between Two Direction Changes

We now extend our study to the case in which a node rests a certain pause time in each waypoint. The total time T' of an RWP period is then composed of a movement transition time T and a pause time T_p, i.e,

$$T' = T + T_p \,. \tag{37}$$

This linear combination of two independent random variables yields an expected value

$$E\{T'\} = E\{T\} + E\{T_p\} \tag{38}$$

and the pdf

$$f_{T'}(\tau') = \int_0^{\tau'} f_T(\tau)\, f_{T_p}(\tau' - \tau)\, d\tau \quad \text{for } \tau' \geq 0\,. \tag{39}$$

The value of $E\{T'\}$ represents the average time between two direction changes. Thus, the direction change frequency is given by $1/E\{T'\}$ in unit 1/s.

Example. We consider a $1000 \times 500\,\text{m}^2$ simulation area. The nodes move according to the RWP model with constant speed $v = 10\,\text{m/s}$ and a pause time taken from an exponential distribution $f_{T_p}(\tau_p) = \mu e^{-\mu\tau_p}$ with a mean pause time $\mu = 10$ s. The expected epoch length is $E\{L\} = 402\,\text{m}$. The expected epoch time is $E\{T'\} = 40.2\,\text{s} + 10\,\text{s} = 50.2\,\text{s}$. If we increase the speed to $v = 20\,\text{m/s}$, we obtain $E\{T'\} = 20.1\,\text{s} + 10\,\text{s} = 30.1\,\text{s}$, i.e., a node is expected to change its direction more frequently. Increasing the area to $1000 \times 1000\,\text{m}^2$ reduces the frequency of direction changes. We then have $E\{T'\} = 62.1$ s (with $v = 10$ m/s) and $E\{T\}' = 36.05$ s (with $v = 20$ m/s).

This example illustrates that, in the RWP model, the mobility metric 'speed' and the size and shape of the area directly influence the mobility metric 'direction change.' In other words, the two metrics 'speed' and 'direction change' cannot be treated as independent input parameters in this model.

4. Spatial Node Distribution

In the previous section, we investigated the distance and time between two consecutive waypoints in the RWP model. These waypoints, which represent the starting and ending points of a node's movement period, are uniformly distributed per definition. In this section, we also take into account the locations that a node visits while moving in a straight line between its waypoints: we study the spatial distribution of nodes resulting from their RWP movement in a rectangular or circular system area $\mathcal{A}$. Again, it is sufficient to regard a single node, because each node moves independently.

Let the random variable $\mathbf{X} = (X, Y)$ denote the Cartesian location of a mobile node in $\mathcal{A}$ at an arbitrary time instant t. A particular outcome of this variable is denoted by $\mathbf{x}$. With this definition, we can express the spatial distribution of a node in terms of the probability density function

$$\begin{aligned} f_{\mathbf{X}}(\mathbf{x}) \;&=\; f_{XY}(x,y) \\ &= \lim_{\delta\to 0} \frac{P\left((x-\frac{\delta}{2} < X \le x+\frac{\delta}{2}) \,\wedge\, (y-\frac{\delta}{2} < Y \le y+\frac{\delta}{2})\right)}{\delta^2} \,. \end{aligned} \tag{40}$$

Note that, in general, a conversion to polar coordinates $R = \sqrt{X^2 + Y^2}$ and $\Phi = \arctan(Y/X)$ yields the joint distribution $f_{R\Phi}(r,\phi) = r \cdot f_{XY}(r\,\cos\phi, r\,\sin\phi)$.

The probability that a given node is located in a certain subarea $\mathcal{A}' \subset \mathcal{A}$ can be computed by integrating $f_{\mathbf{X}}(\mathbf{x})$ over this subarea, i.e.,

$$P(\text{node in } \mathcal{A}') \;=\; P(\mathbf{X} \in \mathcal{A}') \;=\; \iint\limits_{\mathcal{A}'} f_{XY}(x,y)\, dA \,. \tag{41}$$

The differential area element dA is given by $dA = dxdy$ in Cartesian coordinates. The resulting probability $P(\mathbf{X} \in \mathcal{A}')$ can be interpreted as the percentage of time that a given mobile RWP node is located in the subarea $\mathcal{A}'$ during a long–run movement process with many transitions. But it can also be interpreted as the ensemble average if we regard a simulation with many mobile RWP nodes ($n \gg 1$). Then, $E\{n'\} = n\,P(\text{node in } \mathcal{A}')$ denotes the expected number of nodes located in $\mathcal{A}'$ at an arbitrarily chosen time instant.

At the beginning of a simulation, all nodes are typically uniformly distributed, so $f_{\mathbf{X}}(\mathbf{x})$ is given by a uniform distribution over $\mathcal{A}$ at time $t = 0$. However, as it has been observed in [4, 7, 25], this distribution changes as the nodes start to move. This is because the nodes' movement paths tend to go through the center of the system area. For example, a node starting at a waypoint close to the border of the system area clearly finds more destination waypoints in directions toward the center of the area than toward the border. Most likely, it chooses a destination point that requires the node to pass the central area during its next movement period. As time goes on and the nodes perform a number of movement periods, the node distribution becomes more and more non–uniform, with a maximum in the middle of the area and a probability density of $f_{\mathbf{X}}(\mathbf{x}) = 0$ at the borderline. Finally, for a long running time of the movement process, a *stationary distribution* is achieved [3]. In the following, we show this stationary distribution for RWP movement with and without pause time.

4.1. Spatial Distribution without Pause Time

We first regard random waypoint mobility without pause time at the waypoints, i.e., the nodes are continuously moving. Figure 4.1 shows the spatial node distribution $f_{\mathbf{X}}(\mathbf{x})$ of such an RWP movement process obtained through a long–running simulation in a square system area of size $1000 \times 1000\,\mathrm{m}^2$ as well as on a disk of radius $a = 500\,\mathrm{m}$ [3, 6]. In both cases $f_{\mathbf{X}}(\mathbf{x})$ has its maximum at the center of the system area, while the probability of finding a given node close to the border goes to zero. The circular distribution is rotational–symmetric around the center and also the distribution of the square area shows a symmetry. Another important observation is that the distribution is independent of the speed of the nodes.

As suggested in [6], we can use the analytical expression

$$f_{\mathbf{X}}(\mathbf{x}) = f_{XY}(x,y) \approx \frac{36}{a^6}\left(x^2 - \frac{a^2}{4}\right)\left(y^2 - \frac{a^2}{4}\right) \tag{42}$$

to approximate the distribution in a square area of size $a \times a$ defined by $-a/2 \le x \le a/2$ and $-a/2 \le y \le a/2$. As usual, $\iint_{\mathcal{A}} f_{\mathbf{X}}(\mathbf{x})\,dA = 1$ holds. The quality of this approximation is evaluated in detail by Santi and Resta in [24]. In fact, it was shown that (42) approaches very closely the distribution obtained by high–confident simulations. An almost exact equation for the spatial distribution in a square has been derived in [5], and [6] gives an approximation for the circular case. The asymptotic density function on a line segment $[0, a]$ is given by

$$f_X(x) = -\frac{6}{a^3}\,x^2 + \frac{6}{a^2}\,x \tag{43}$$

for $0 < x < a$, and 0 otherwise [5, 6].

4.2. Spatial Distribution with Pause Time

Let us now move on to study RWP nodes that are also allowed to pause a certain amount of time τ_p at their destination points. The resulting spatial node distribution $f_{\mathbf{X}}(\mathbf{x})$ is then given by the superposition of two distinct components, namely a pause component and mobility component. Both components are probability density functions weighted by the probability that a node pauses or moves, respectively [24]. Let p_p denote the probability that a given node pauses at a randomly chosen time. We can then set up:

$$f_{\mathbf{X}}(\mathbf{x}) \;=\; \underbrace{p_p\, f_{\mathbf{X},p}(\mathbf{x})}_{\text{pause component}} \;+\; \underbrace{(1-p_p)\; f_{\mathbf{X},m}(\mathbf{x})}_{\text{mobility component}}\;. \tag{44}$$

The pdf $f_{\mathbf{X},p}(\mathbf{x})$ represents the spatial distribution of all nodes that are currently pausing at a destination point. Since the destination points are chosen from a uniform distribution,

$f_{\mathbf{X},p}(\mathbf{x})$ is also uniform. Thus,

$$f_{\mathbf{X},p}(\mathbf{x}) = \begin{cases} \frac{1}{\|\mathcal{A}\|} & \mathbf{x} \in \mathcal{A} \\ 0 & \text{else} \end{cases} . \quad (45)$$

The pdf $f_{\mathbf{X},m}(\mathbf{x})$ of the mobility component represents the spatial distribution of all moving nodes. It is thus given by the results of Section 4.1, i.e., the right hand side of (42) for a square.

Recall that τ_i and $\tau_{p,i}$ denote the time that a node moves or pauses, respectively, during an RWP period i. The pause probability p_p is given by the percentage of time that a node pauses during a long–running process. Assuming that each node pauses a fixed time period τ_p at each waypoint i (i.e., $T_{p,i} = \tau_p \; \forall i$) we have [24]

$$p_p = \lim_{m\to\infty} \frac{\sum_{i=1}^{m} \tau_{p,i}}{\sum_{i=1}^{m} (\tau_{p,i} + \tau_i)} = \frac{\tau_p}{\tau_p + E\{T\}} . \quad (46)$$

with the expected transition time $E\{T\} = \lim_{m\to\infty} \frac{1}{m} \sum_{i=1}^{m} \tau_i$ given in Section 3.5. If we regard a generalization in which the pause time is taken from a pdf $f_{T_p}(\tau_p)$ with an expected value $E\{T_p\}$, the pause probability can be calculated by

$$p_p = \frac{E\{T_p\}}{E\{T_p\} + E\{T\}} . \quad (47)$$

Applying the values of p_p in (44) allows for the computation of the node distribution in a variety of scenarios. Table 4.2 shows some mappings between the pause probability p_p and expected pause time $E\{T_p\} = \frac{p_p}{1-p_p} \cdot \frac{E\{L\}}{v}$ in a square system area of size $a \times a$.

5. Movement Direction

One of the major reasons why the spatial node distribution resulting from the RWP model is non–uniform is the fact that nodes take a non–uniformly distributed direction angle at the beginning of each movement period i.[1] The probability density function of this angle is determined by the shape of the system area and the starting waypoint $\mathbf{p}$ of the node. Let us investigate this issue in more detail for a one–dimensional line segment and a circular system area. All angles are defined in radian notation.

5.1. One–Dimensional Line

We first consider a one–dimensional line segment $[0, a]$. The random variable representing the direction of a node is denoted by Γ and a specific value is γ. A node located at a given

[1] Note that other stochastic mobility models which directly choose a destination direction rather than a destination point and allow a bounce back or wrap–around behavior at the border of the system area are able to achieve a uniform spatial distribution (see, e.g., [3]).

waypoint $\mathbf{P} = x$, with $0 \leq x \leq a$, chooses a new destination point and then moves either to the right (direction $\Gamma = 0$) or to the left (direction $\Gamma = \pi$).

It is straightforward to observe that a certain direction is chosen with high probability, if many potential destination points lie in this direction. Since the destination points are taken from a uniform distribution, we obtain $P(\Gamma = 0 \,|\, \mathbf{P}{=}x) = \frac{a-x}{a}$ and $P(\Gamma{=}\pi \,|\, \mathbf{P}{=}x) = \frac{x}{a}$, and therefore

$$f_\Gamma(\gamma \mid \mathbf{P} = x) = \frac{a-x}{a}\,\delta(\gamma) + \frac{x}{a}\,\delta(\gamma - \pi)\,, \tag{48}$$

where $\delta(\cdot)$ again denotes the Dirac delta function.

5.2. Circular Area

In a circular area it is useful to employ polar coordinates for the location of the starting waypoint. We use P_r and P_ϕ as random variables, and r and ϕ for particular values of theses variables. The definition of a node's movement direction γ (random variable Γ) in a two–dimensional system space is shown in Figure 5.2. It denotes the angle between a horizontal line and the current movement vector of the node, where $0 \leq \gamma < 2\pi$. In each waypoint a node 'chooses' such a direction angle whose value remains constant during one movement period. Unfortunately, the outcome of this angle depends on both polar coordinates of the starting waypoint. Thus, as shown in Fig. 5.2, we introduce in each waypoint a second, alternative direction angle denoted as θ (random variable Θ). This angle is defined in a way that $\Theta{=}0$ for movement transitions going through the center of the circular system area. Again we have $0 \leq \Theta < 2\pi$. A major advantage of this definition is that the outcome of Θ is independent of the polar angle ϕ of the starting waypoint. On the other hand, it is only convenient to use θ in the waypoints. It can then be directly mapped to γ for given ϕ. If we define that counterclockwise angles count positive, we have

$$\gamma = \theta + \phi + \pi\,. \tag{49}$$

Our aim is now to derive first the distribution of Θ for a given starting waypoint $\mathbf{P}{=}\mathbf{p}$. This conditional distribution is denoted by $f_\Theta(\theta \,|\, \mathbf{P}{=}\mathbf{p})$. Later, we transform this distribution to $f_\Gamma(\gamma \,|\, \mathbf{P}{=}\mathbf{p})$. Both probability density functions give us information about the direction that a node takes when it starts at a waypoint with known location $\mathbf{p}$. Finally, we integrate over all possible starting points in the circle to obtain the unconditional pdf $f_\Theta(\theta)$.

Without any calculation, we can already make the following four statements about the pdf of the direction Θ in a given waypoint. First, it is independent of the polar angle ϕ of the starting waypoint, i.e., $f_\Theta\,(\theta \,|\, \mathbf{P}{=}\mathbf{p}) = f_\Theta(\theta \mid P_r = r)$. Thus, in the following, we just write $f_\Theta(\theta \,|\, r)$ for simplicity of notation. Second, the direction is uniformly distributed for nodes starting at $P_r{=}0$, i.e.,

$$f_\Theta(\theta \,|\, 0) = \frac{1}{2\pi} \quad \text{for} \quad 0 \leq \theta < 2\pi\,.$$

Third, for all $P_r{\neq}0$, the highest value in the pdf is achieved for $\Theta{=}0$, the lowest for $\Theta{=}\pi$:

$$\max\left(f_\Theta(\theta \mid r)\right) \;=\; f_\Theta(0 \mid r) \qquad \text{for} \quad 0 < r < a$$

$$\min\left(f_\Theta(\theta \mid r)\right) \;=\; f_\Theta(\pi \mid r) \qquad \text{for} \;\; 0 < r < a.$$

Last but not least, the pdf shows the symmetry

$$f_\Theta(\theta - \Delta\theta \mid r) \;=\; f_\Theta(\theta + \Delta\theta \mid r) \quad \forall \Delta\theta,\, \forall r\,.$$

In the following we derive $f_\Theta(\theta \,|\, r)$. Let $P(\Theta \le \theta \,|\, r)$ denote the probability that a node at a given waypoint (r, ϕ) takes a movement angle Θ that is lower than a certain value θ. Clearly, a node takes such a direction, if it chooses a destination point within the area spanned by $0 \le \Theta < \theta$ (see Fig. 5.2). Let us denote this subarea by $\mathcal{A}_\theta$. The probability that a destination point is chosen in $\mathcal{A}_\theta$ is given by the area integral of the spatial pdf $f_{P_x P_y}(x, y)$ over this subarea. Since the destination waypoints are chosen from a uniform spatial distribution, we can set up

$$P(\Theta \le \theta \mid r) = \iint\limits_{\mathcal{A}_\theta} \frac{1}{a^2\pi}\, dA \;=\; \frac{\|\mathcal{A}_\theta\|}{a^2\pi}\,, \tag{50}$$

with the area element given by $dA = dx\,dy = r\,dr\,d\phi$. For given r, we define the length $l_{max}(\theta')$ as the maximum possible transition length of a node at r in the direction θ' (see Fig. 5.2). Note that this length is independent of the polar angle ϕ. Using the law of cosines for θ' in the triangle with sides $\{l_{max}(\theta'), a, \text{and } r\}$, we obtain a quadratic equation for $l_{max}(\theta')$, whose solution is

$$l_{max}(\theta') = r\cos\theta' + \sqrt{a^2 - r^2 \sin^2\theta'}\,. \tag{51}$$

The area size $\|\mathcal{A}_\theta\|$ can now be computed by

$$\|A_\theta\| \;=\; \int_{\theta'=0}^{\theta} \int_{r'=0}^{l_{max}(\theta')} r'\,dr'\,d\theta' \;=\; \int_{\theta'=0}^{\theta} \frac{1}{2}\, l_{max}^2(\theta')\,d\theta'\,, \tag{52}$$

The desired pdf $f_\Theta(\theta \,|\, r)$ is given by the derivate of $P(\Theta \le \theta \,|\, r)$ with respect to θ, which gives

$$f_\Theta(\theta \mid r) \;=\; \frac{\partial}{\partial\,\theta}\, P(\Theta \le \theta \,|\, r) \;=\; \frac{1}{a^2\pi}\,\frac{1}{2}\, l_{max}^2(\theta)\,. \tag{53}$$

We can conclude with the following result.

Result. A node moves in a circular area of radius a according to the RWP model. At the beginning of each movement period, it chooses a new destination waypoint. The pdf of the direction Θ from the old waypoint (r, ϕ) toward the new waypoint is

$$f_\Theta(\theta \mid r) = \frac{1}{2\pi}\left(\frac{r}{a}\cos\theta + \sqrt{1 - \frac{r^2}{a^2}\sin^2\theta}\right)^2\,. \tag{54}$$

The plot of this pdf is shown in Figures 5.2 and 5.2 for various r. Let us briefly discuss this distribution. We can say that the above statements on the symmetry and maxima/minima hold. Furthermore, we observe that the function just depends on the ratio r/a. Nodes located at the border of the area, i.e., r close to a, have a high probability of taking θ around 0, i.e., they tend to move toward the center of the disk. In the extreme case $r \to a$, we have $f_\Theta(\theta \,|\, r) = 0$ for $\frac{\pi}{2} < \theta < \frac{3}{2}\pi$. This effect decreases as a node starts closer to the center of the disk, until a uniform direction distribution is achieved for $r = 0$.

As mentioned above, we now convert our result to the random variable Γ. This yields

$$f_\Gamma(\gamma \,|\, (r, \phi)) \;=\; f_\Theta(\gamma - \phi - \pi \,|\, r) \tag{55}$$

and

$$E\{\Gamma\} = \phi + \pi\;. \tag{56}$$

Finally, we calculate from the conditional pdf $f_\Theta(\theta \,|\, r)$ the unconditional pdf $f_\Theta(\theta)$. This distribution informs us about the direction θ that a node chooses at an arbitrary waypoint, if we do not know the location of this waypoint. Weighting $f_\Theta(\theta \,|\, r)$ with the (uniform) spatial distribution of the waypoints and integrating over all possible locations yields

$$\begin{aligned} f_\Theta(\theta) \;&=\; \int_0^{2\pi}\int_0^a f_\Theta(\theta \,|\, r)\,\frac{1}{a^2\pi}\,r\,dr\,d\phi \\ &=\; \frac{1}{4\pi\,|\sin^3\theta|}\Big[\,|\sin\theta|\,\Big(-2\cos^4\theta - 2\cos^3\theta\,|\cos\theta| + \cos^2\theta + \cos\theta\,|\cos\theta| + 1\Big) \\ &\qquad + \arcsin\left(|\sin\theta|\right)\,\cos\theta\,\Big]\,. \end{aligned} \tag{57}$$

The plot of $f_\Theta(\theta)$ is also shown in Figure 5.2. Clearly, the most frequent direction is $\Theta = 0$ while $\Theta = \pi$ is very unlikely. In fact, the probability that a node takes its direction within the interval $\left[\frac{\pi}{2}, \frac{3\pi}{2}\right]$ is only 12.5 %, whereas in 61.4 % of all movement transitions a node moves toward the central region of the area with $-\frac{\pi}{4} \le \Theta \le \frac{\pi}{4}$.

6. Cell Change Rate

We now regard RWP movements on a system area that is structured by a grid into a set of cells. We are then interested in the corresponding cell change rate of an RWP node, defined as the number of cell changes per unit time. There are two reasons for studying the cell change rate: first, our major motivation is that some network services in a mobile ad hoc network, like the Grid Location Service [21], assume that the system area shows a grid structure and nodes have to send some signaling messages whenever they move from one cell to another one. A second reason is that the RWP model can also be applied to model a node's mobility behavior in a cellular network, e.g., in a wireless LAN system, where the cells represent the radio coverage of the base stations. In both scenarios, the amount of signaling traffic is related to the cell change rate.

For this analysis we consider a rectangular system area of size $a \times b$. This area is divided into $\alpha \times \beta$ rectangular cells of equal size. Figure 6 gives an example with $a = b$ and $\alpha = \beta = 3$.

A cell change event occurs if a node crosses the boundary line between two cells. Clearly, the assumption of rectangular non–overlapping cells is an idealization for the second scenario described above. Real radio cells have various shapes and are overlapping. Moreover, the cell change rate depends, for example, on the handover decision and location update algorithm. However, since stating an expression that covers all the particular cases is not feasible, we study the simplest but most general case, the grid scenario.

6.1. Cell Changes per Transition

Our first aim is to analyze how many cell boundaries a node crosses on average during one movement transition. The corresponding random variable is denoted by C. By denoting c_i the outcome of this random variable in transition i, we can write

$$E\{C\} = \lim_{m\to\infty} \frac{1}{m} \sum_{i=1}^{m} c_i \,. \tag{58}$$

The outcome of C is a deterministic function of the random starting waypoint $\mathbf{P}_1 = (P_{x1}, P_{y_1})$ of a node and its next destination waypoint $\mathbf{P}_2 = (P_{x2}, P_{y_2})$. In fact, in our grid structure, the number of cell changes depends only on the cell positions of these two independent waypoints. In the following, the horizontal position of a cell is indexed by $\xi = 1, 2, \ldots, \alpha$ and its vertical position by $\psi = 1, 2, \ldots, \beta$. The cell position of a point can thus be written as (ξ, ψ). A point at (x, y) has cell position $\xi = \lceil \frac{x\alpha}{a} \rceil$ and $\psi = \lceil \frac{y\beta}{b} \rceil$, where $\lceil x \rceil$ denotes the ceiling function giving the smallest integer that is larger then or equal to x. If a node moves from a waypoint in cell (ξ_1, ψ_1) to a waypoint in cell (ξ_2, ψ_2), the number of cell changes is given by the Manhattan distance (l_1 norm) between the cells, i.e.,

$$c(\xi_1, \psi_1, \xi_2, \psi_2) = |\xi_1 - \xi_2| + |\psi_1 - \psi_2| \,. \tag{59}$$

The expected value of C can then be computed by the average of $c(\xi_1, \psi_1, \xi_2, \psi_2)$ over all possible cell pairs:

$$E\{C\} = \frac{1}{\alpha^2\beta^2} \sum_{\xi_1=1}^{\alpha} \sum_{\psi_1=1}^{\beta} \sum_{\xi_2=1}^{\alpha} \sum_{\psi_2=1}^{\beta} c(\xi_1, \psi_1, \xi_2, \psi_2) \,. \tag{60}$$

Example. We give a simple example for a scenario with 3×3 cells (see Fig. 6). A movement transition from cell $(1,1)$ to $(1,1)$ yields no cell change, a transition from $(1,2)$ to $(1,1)$ or vice versa yields $c = 1$ cell change, between $(2,2)$ to $(1,1)$ we obtain $c = 2$, and so on. Using Equation (59) in (60) with $\alpha = \beta = 3$ yields $E\{C\} = 1.778$ cell changes per transition.

Let us make some more comments on the above result. We observe that if we use the Manhattan distance $|\xi_1 - \xi_2| + |\psi_1 - \psi_2|$, the expected number of changes can be expressed as a sum $E\{C\} = E\{C_\xi\} + E\{C_\psi\}$, where $C_\xi = |\xi_1 - \xi_2|$ and $C_\psi = |\psi_1 - \psi_2|$ represent the number of horizontal or vertical cell changes, respectively. Moreover we note that an upper bound for $E\{C\}$ is given by $\frac{1}{3}(\alpha+\beta)$. For many cells, i.e., high α and β, this bound can be used as a good approximation for $E\{C\}$. In other words,

$$E\{C\} \;=\; \frac{1}{3}\,(\alpha+\beta)-\epsilon\,,\ \text{with small } \epsilon>0\,. \tag{61}$$

The relative error of this approximation, $\epsilon/E\{C\}$, is lower than $+1\,\%$ if $\alpha>10\wedge\beta>10$, and it decreases for increasing α and β. In the asymptotic case, we have $\lim_{\alpha,\beta\to\infty}\epsilon=0$.

Using the Manhattan distance metric, the movement of a node to a diagonally adjacent cell is always counted as two cell changes. We do this because the probability that such a movement transition goes directly through a single grid point goes to zero. Thus, the set of all transitions crossing a single grid point is a 'null set.'

6.2. Cell Change Rate

We are now interested in the average number of cell changes of an RWP node per unit time, denoted by $E\{C_t\}$. This value can be interpreted as the 'cell change rate' or 'cell change frequency' and serves as a good measure for the degree of mobility of a node or a certain scenario.

The expected number of cell changes per unit time is given by the sum of all cell changes that occur in a long run simulation divided by the entire simulation time. We can therefore apply our results from the number of cell changes per movement transition and the transition time (Section 3). We write:

$$E\{C_t\} \;=\; \lim_{m\to\infty}\frac{\sum_i^m C_i}{\sum_i^m T_i} \;=\; \frac{E\{C\}}{E\{T\}}\,. \tag{62}$$

Example. An RWP node moves on a square system area $\mathcal{A}$ consisting of $\alpha\times\alpha$ cells, each cell with a side length of 250 m. In each waypoint it chooses a new speed V from a uniform distribution between v_{min} and v_{max}. There is no pause time in the destination points. We now investigate the cell change rate as the number of cells α^2 increases. Using $\alpha\times\alpha$ cells of fixed side length 250 m, the size of the system area is given by $\|\mathcal{A}\| = 62500\,\alpha^2\,\mathrm{m}^2$. Using $E\{C_t\}=E\{C\}/E\{T\}$ with (59,60) and (33), we computed the number of cell changes per minute for four different pairs of v_{min} and v_{max}. The result is shown in Figure 6.2.

For each speed distribution we obtain a different cell change rate. The dependency on the number of cells shows an asymptotic behavior for an increasing system area: if α is small the cell change rate increases significantly, but if α is large enough it is almost independent on α and converges smoothly toward an asymptotic value. This is because for a large system area, the maximum number of cell crossings per transition is limited only by the speed of the node. The asymptotic cell change rate can be computed using (61) with $\epsilon=0$. It represents a metric for the node mobility of a given scenario. For example, a node with a uniform speed distribution within $3\ldots8$ m/s has a higher cell change rate than a node with a uniform speed within $1\ldots10$ m/s, although both node types have the same average speed $E\{V\}=5.5$ m/s.

7. Related Work

Related work by other authors on the analysis of the random waypoint model has been published quite recently in [7, 10, 24, 26, 27]. Blough, Resta, and Santi [7] performed a simulation–based study in order to test for which parameters the spatial node distribution resulting from the RWP model and a Brownian motion–like model can be approximated by the uniform distribution. As mentioned above, the same group of authors investigated the impact of pause time on the node distribution [24].

Furthermore, Song and Miller [27] studied by means of OPNET simulations how the RWP parameters influence a node's 'mobility factor.' This factor is defined in that paper as the average change of distance of a node to all other nodes per unit time. Another paper that contains simulation results on various RWP issues is by Chu and Nikolaidis [10]. Finally, the paper [5], by Bettstetter, Resta, and Santi derives in an analytical manner an almost exact equation for the spatial node distribution of the RWP model in a square area.

8. Conclusions

This paper presented a mathematical analysis of some stochastic properties of the well–known random waypoint mobility model. After giving a formal description of this model in terms of a discrete–time stochastic process, we investigated (a) the length and duration of a movement between two waypoints, (b) the resulting spatial node distribution and its dependence on the pause time, (c) the chosen direction angle at the beginning of a movement transition, and (d) the number of cell changes for an RWP model used in a cellular–structured network. These results give a deeper understanding on how this model behaves in simulation–based analysis of mobile wireless networks.

In particular, our equations for the movement transition time as well as the cell change rate enable us to make a statement about the 'degree of mobility' of a certain simulation scenario. Such a mobility metric is needed if we want to compare simulation results made with the RWP model and a different model and to identify the influence of mobility on simulation results. We have shown that the time between two direction changes is determined by the speed of the nodes and size and shape of the area. The knowledge of the spatial node distribution is essential for simulation–based investigations on interference between nodes, medium access control, and connectivity issues, to name a few. Finally, the derived direction distribution explains in an analytical manner the effect reported in [4] and [25] that nodes tend to move back to the middle of the area. Being unaware of this behavior may lead to misinterpretation of simulation results.

The methods employed in this paper can also be applied to other models, in order to derive appropriate measures describing stochastic mobility in a precise and meaningful manner.

Acknowledgements

The authors would like to thank Bhaskar Sengupta and an anonymous reviewer for useful comments on preliminary versions of this paper. Special thanks goes to Paolo Santi for several discussions on the random waypoint model. Part of this research was funded by the Deutsche Forschungsgemeinschaft (DFG).

References

1. Glomosim. http://pcl.cs.ucla.edu/projects/glomosim/, Nov. 2002.
2. The network simulator - ns-2. http://www.isi.edu/nsnam/ns/, June 2002.
3. C. Bettstetter. Mobility modeling in wireless networks: Categorization, smooth movement, and border effects. *ACM Mobile Comp. and Comm. Review*, 5(3), July 2001.
4. C. Bettstetter. Smooth is better than sharp: A random mobility model for simulation of wireless networks. In *Proc. ACM Intern. Workshop on Modeling, Analysis, and Sim. of Wireless and Mobile Systems (MSWiM)*, Rome, Italy, July 2001.
5. C. Bettstetter, G. Resta, and P. Santi. The node distribution of the random waypoint mobility model for wireless ad hoc networks. Technical report, Technische Universität München, Institute of Communication Networks, Munich, Germany, Sept. 2002.
6. C. Bettstetter and C. Wagner. The spatial node distribution of the random waypoint mobility model. In *Proc. German Workshop on Mobile Ad Hoc Networks (WMAN)*, Ulm, Germany, Mar. 2002.
7. D. M. Blough, G. Resta, and P. Santi. A statistical analysis of the long-run node spatial distribution in mobile ad hoc networks. In *Proc. ACM Intern. Workshop on Modeling, Analysis, and Sim. of Wireless and Mobile Systems (MSWiM)*, Atlanta, GA, Sept. 2002.
8. L. Briesemeister and G. Hommel. Role–based multicast in highly mobile but sparsely connected ad hoc networks. In *Proc. Workshop on Mobile Ad Hoc Networking and Computing (MobiHoc)*, Boston, MA, Aug. 2000.
9. J. Broch, D. A. Maltz, D. B. Johnson, Y.-C. Hu, and J. Jetcheva. A performance comparison of multi-hop wireless ad hoc network routing protocols. In *Proc. ACM Intern. Conf. on Mobile Comp. and Netw. (MobiCom)*, Dallas, TX, Oct. 1998.
10. T. Chu and I. Nikolaidis. On the artifacts of random waypoint simulations. In *Proc. Intern. Workshop on Wired/Wireless Internet Comm. (WWIC)*, Las Vegas, NV, June 2002.
11. S. R. Das, C. E. Perkins, and E. M. Royer. Performance comparison of two on-demand routing protocols for ad hoc networks. In *Proc. IEEE Infocom*, Tel Aviv, Israel, 2000.
12. B. Ghosh. Random distances within a rectangle and between two rectangles. *Bull. Calcutta Math. Soc.*, 43:17–24, 1951.
13. R. A. Guérin. Channel occupancy time distribution in a cellular radio system. *IEEE Trans. Vehicular Techn.*, 36(3), Aug. 1987.
14. J. M. Hammersley. The distribution of distance in a hypersphere. *Annals of Math. Statistics*, 21:447–452, 1950.
15. D. Heidemann. Ein mathematisches Modell des Verkehrsflusses. *Schriftenreihe des Instituts für Verkehrswesen, Universität Karlsruhe, Germany*, 42, 1989.
16. D. Helbing. Traffic and related self–driven many-particle systems. *Reviews of Modern Physics*, 73(4):1067–1141, Oct. 2001.
17. G. Holland and N. H. Vaidya. Analysis of TCP performance over mobile ad hoc networks. In *Proc. ACM Intern. Conf. on Mobile Comp. and Netw. (MobiCom)*, Seattle, WA, Aug. 1999.

18. X. Hong, M. Gerla, G. Pei, and C.-C. Chiang. A group mobility model for ad hoc wireless networks. In *Proc. ACM Intern. Workshop on Modeling, Analysis, and Sim. of Wireless and Mobile Systems (MSWiM)*, Seattle, WA, 1999.
19. E. Jugl. *Mobilitätsmodellierung und Einflüsse auf Systemparameter von Mobilfunksystemen*. PhD thesis, TU Ilmenau, 2000.
20. D. Lam, D. C. Cox, and J. Widom. Teletraffic modeling for personal communication services. *IEEE Communications*, 35(2):79–87, Oct. 1997.
21. J. Li, J. Jannotti, D. D. Couto, D. Karger, and R. Morris. A scalable location service for geographic ad hoc routing. In *Proc. ACM Int. Conf. on Mobile Computing and Networking (MobiCom)*, Boston, MA, 2000.
22. J. G. Markoulidakis, G. L. Lyberopoulos, D. F. Tsirkas, and E. D. Sykas. Mobility modeling in third–generation mobile telecommunication systems. *IEEE Pers. Comm.*, Aug. 1997.
23. A. Papoulis and S. U. Pillai. *Probability, random variables, and stochastic processes*. McGraw–Hill, 4th edition, 2002.
24. G. Resta and P. Santi. An analysis of the node spatial distribution of the random waypoint model for ad hoc networks. In *Proc. ACM Workshop on Principles of Mobile Computing (POMC)*, Toulouse, France, Oct. 2002.
25. E. M. Royer, P. M. Melliar-Smith, and L. E. Moser. An analysis of the optimum node density for ad hoc mobile networks. In *Proc. IEEE Intern. Conf. on Comm. (ICC)*, Helsinki, Finland, June 2001.
26. P. Santi and D. M. Blough. An evaluation of connectivity in mobile wireless ad hoc networks. In *Proc. IEEE Intern. Conf. on Dependable Systems and Networks (DSN)*, Washington, DC, June 2002.
27. J. Song and L. E. Miller. Empirical analysis of the mobility factor for the random waypoint model. In *Proc. OPNETWORK*, Washington, DC, Aug. 2002.
28. E. Weisstein, editor. *Eric Weisstein's World of Mathematics*, chapter Probability and Statistics: Statistical Distributions. Wolfram Research, online edition, Dec. 2002.
29. M. M. Zonoozi and P. Dassanayake. User mobility modeling and characterization of mobility patterns. *IEEE Journal on Sel. Areas in Comm.*, 15(7):1239–1252, Sept. 1997.

Authors' Vitae

Christian Bettstetter

Christian Bettstetter, born in 1973, is a research staff member and Ph.D. candidate at the Institute of Communication Networks at Technische Universität München (TUM), Germany. He graduated from TUM in electrical engineering and information technology (Dipl.–Ing.) in 1998. His interests are in the field of mobile and IP–based communication networks, where his main research area is wireless ad hoc and sensor networking. Christian published over 30 technical articles and co–authored the Wiley book 'GSM: Switching, services and protocols.' E–mail: Christian.Bettstetter@ei.tum.de

Hannes Hartenstein

Hannes Hartenstein received the master's degree in mathematics in 1995 and the Ph.D. degree in computer science in 1998, both from Albert–Ludwigs–Universität, Freiburg, Germany. He was Erasmus Scholar with the University of East Anlia, Norwich, U.K., in 1991–1992

and received the Capocelli Award from the IEEE Data Compression Conference 1997 (with M. Ruhl) for the paper 'Optimal fractal coding is NP–hard.' He joined NEC Network Laboratories Europe in 1999 and is now a Senior Research Staff Member with the Mobile Internet Group. He currently focuses on IP–based mobility management as well as on ad hoc routing. His research interests include mobile communications, networking, multimedia, image/video processing, and theoretical computer science.
E–mail: Hannes.Hartenstein@ccrle.nec.de

Xavier Pérez–Costa
Xavier Pérez–Costa received his master's degree in telecommunications engineering from the Polytechnic University of Catalonia (UPC) in 2000. He started working at NEC Network Laboratories Europe Ltd in Heidelberg on a topic related to QoS in IEEE 802.11 wireless LAN and is now a Research Staff Member within the Mobile Internet Group focusing on IP–based mobility management. His main research interests are communication networks, mobile networking, and quality of service.
E–mail: Xavier.Perez-Costa@ccrle.nec.de

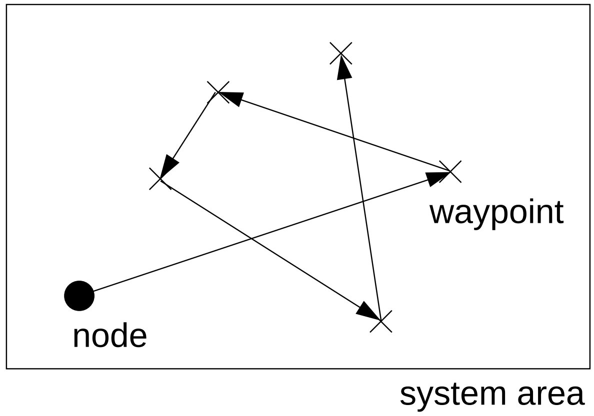


Figure 1. Illustration of random waypoint movement

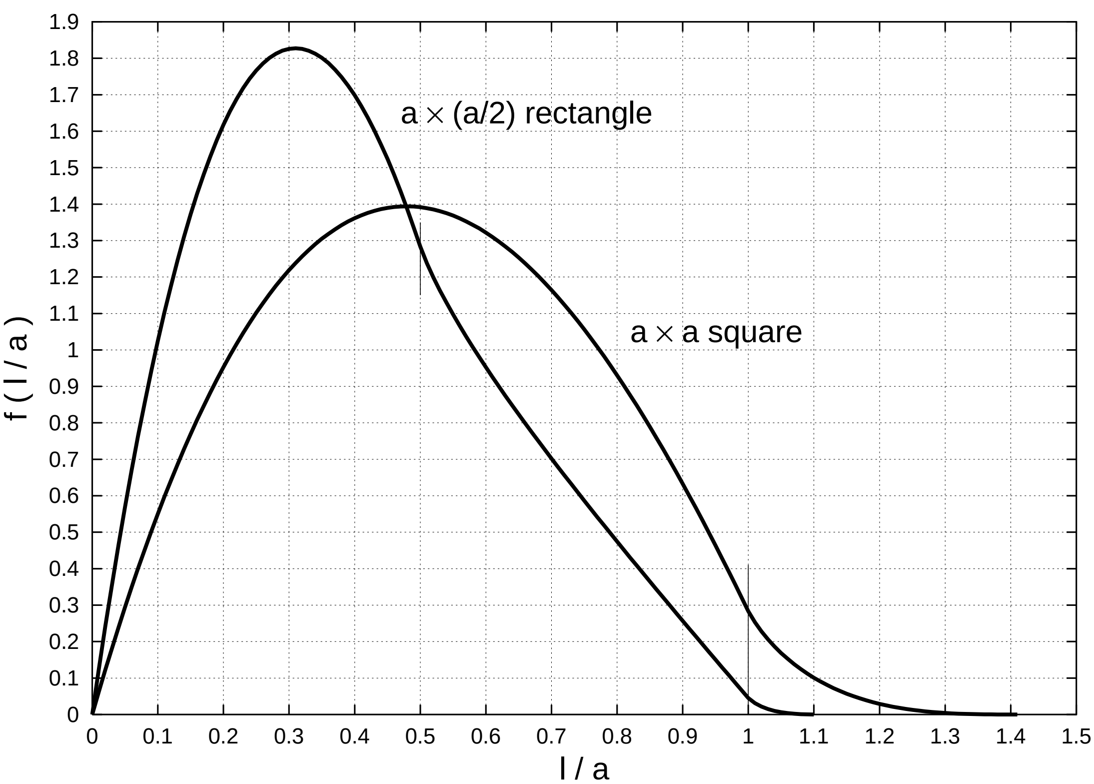


Figure 2. Pdf of transition length of RWP nodes in a rectangle

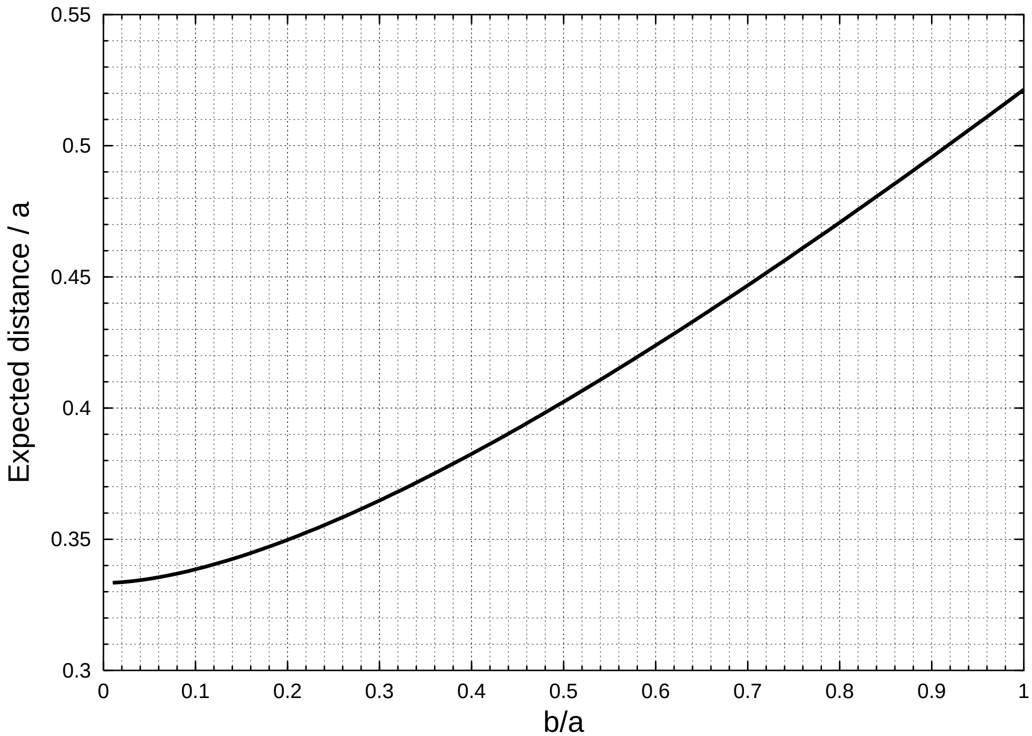


Figure 3. Expected transition length of RWP nodes within an $a \times b$ rectangle

Table I. Mapping between pause probability and expected pause time (square area)

p_p	0	0.1	0.161	0.2	0.5	0.658	0.9	0.951	1.0
$\frac{v}{a} E\{T_p\}$	0	0.058	0.1	0.130	0.521	1.0	4.689	10	∞

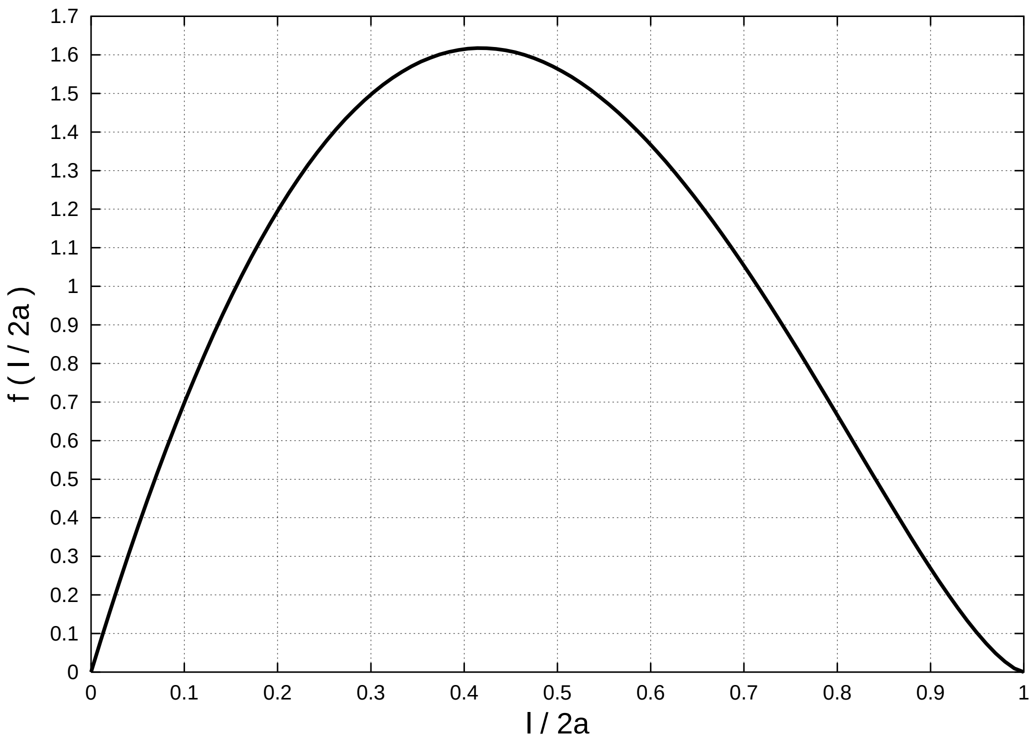


Figure 4. Pdf of transition length of RWP nodes on a disk of radius a

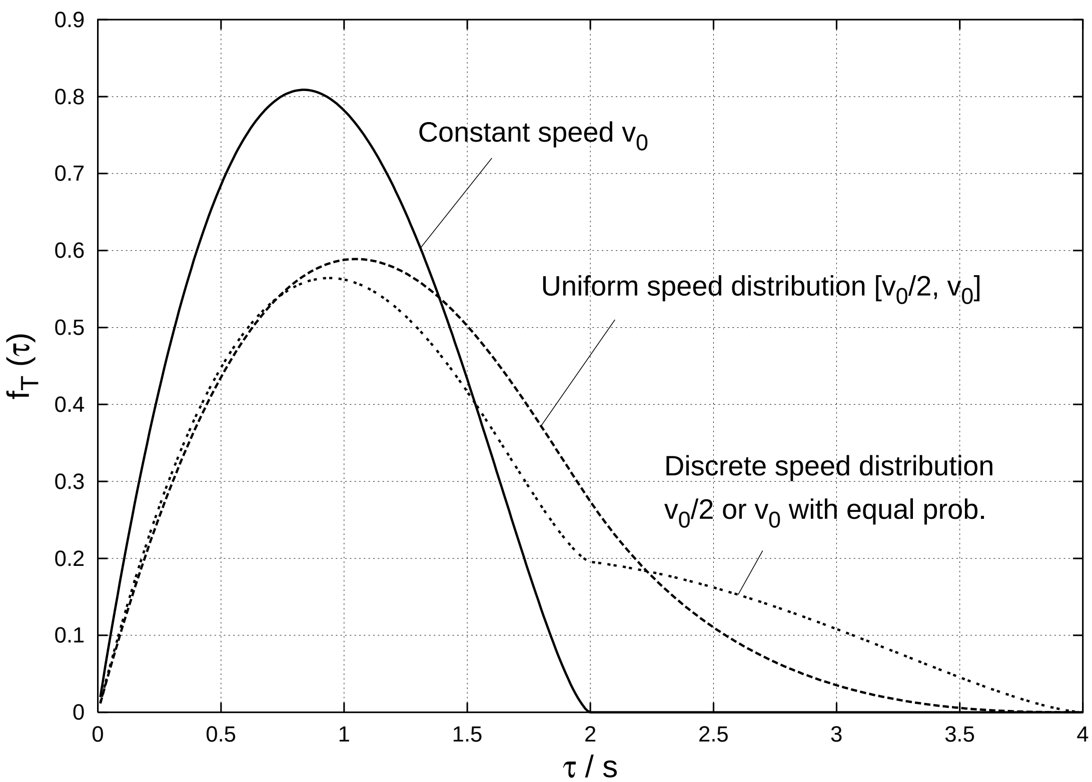
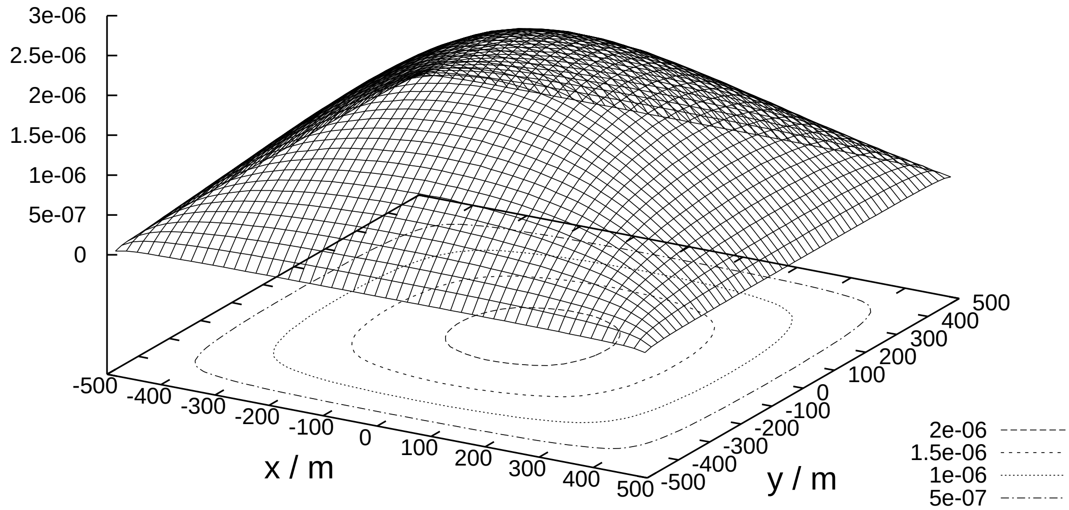
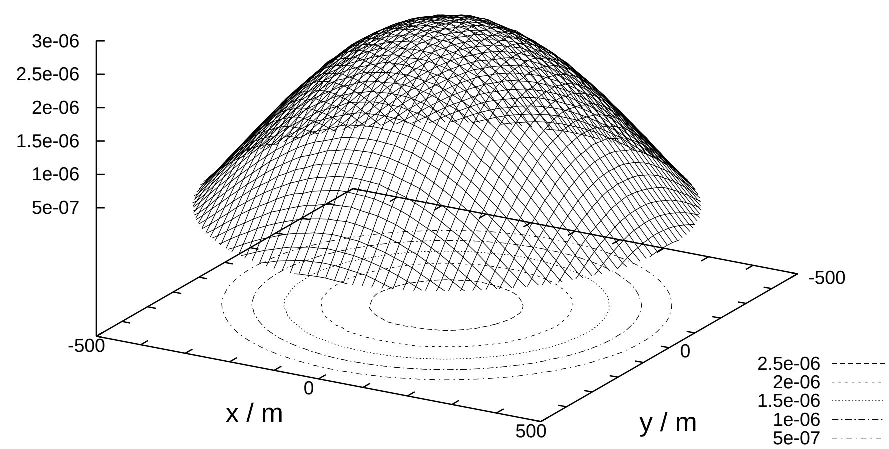


Figure 5. Pdf of transition time of RWP nodes on a disk ($a = 1\,\mathrm{m}$, $v_0 = 1\,\mathrm{m/s}$)



a. Square system area



b. Circular system area

Figure 6. Spatial node distribution resulting from the RWP model (simulation results)

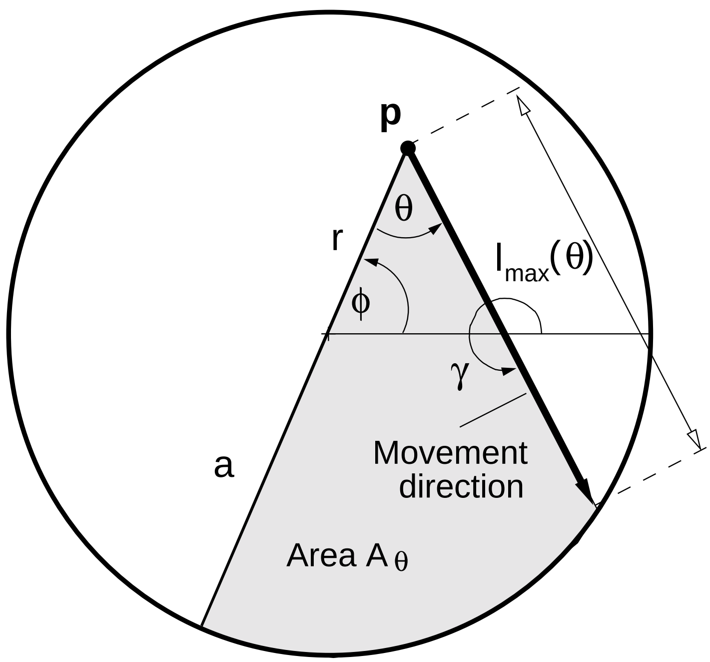


Figure 7. Definition of direction angles

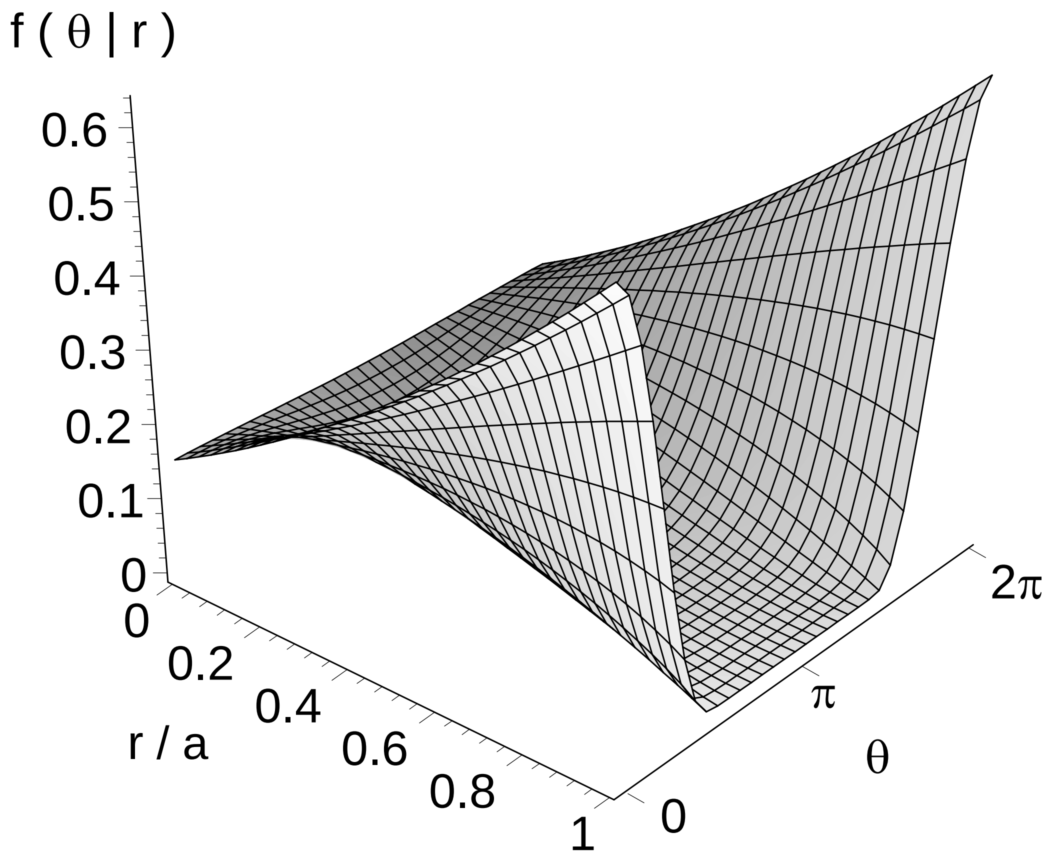


Figure 8. Distribution of movement direction

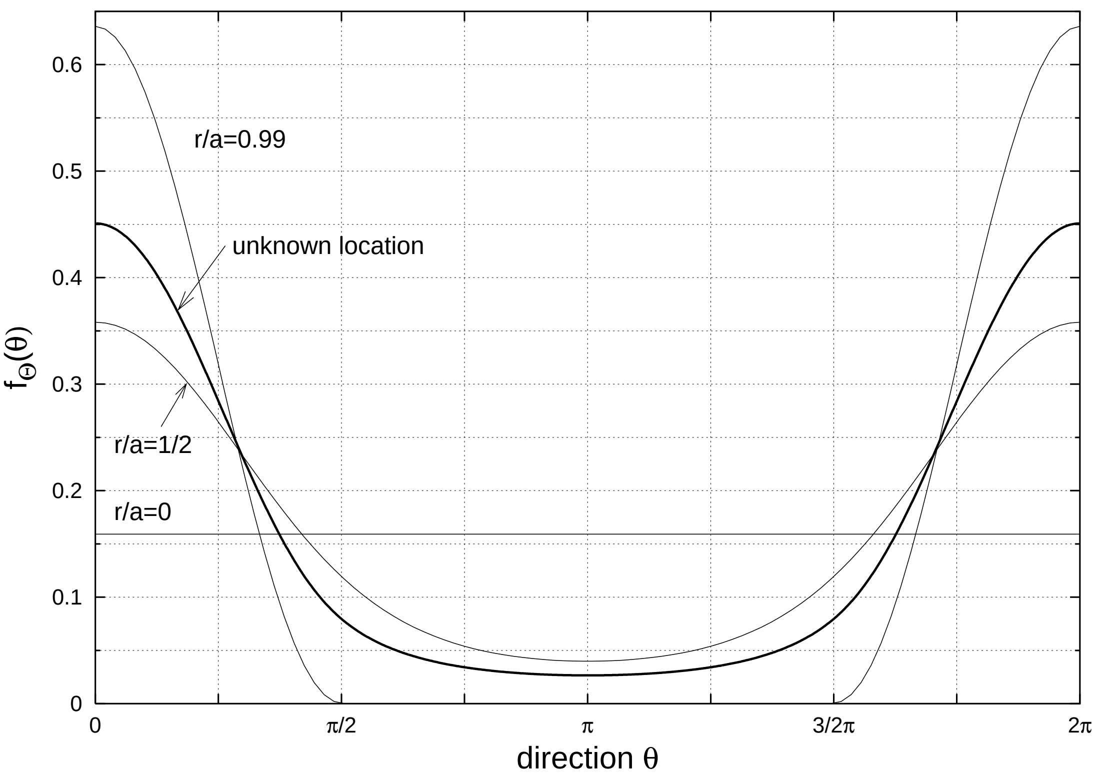


Figure 9. Distribution of movement direction

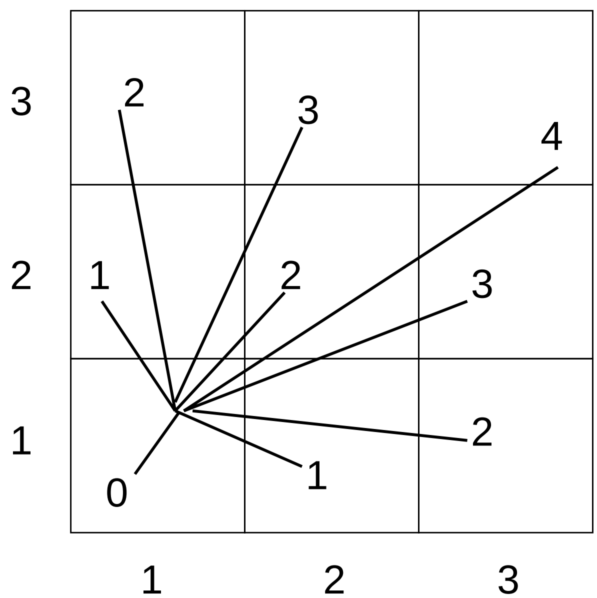


Figure 10. Cell changes per transition

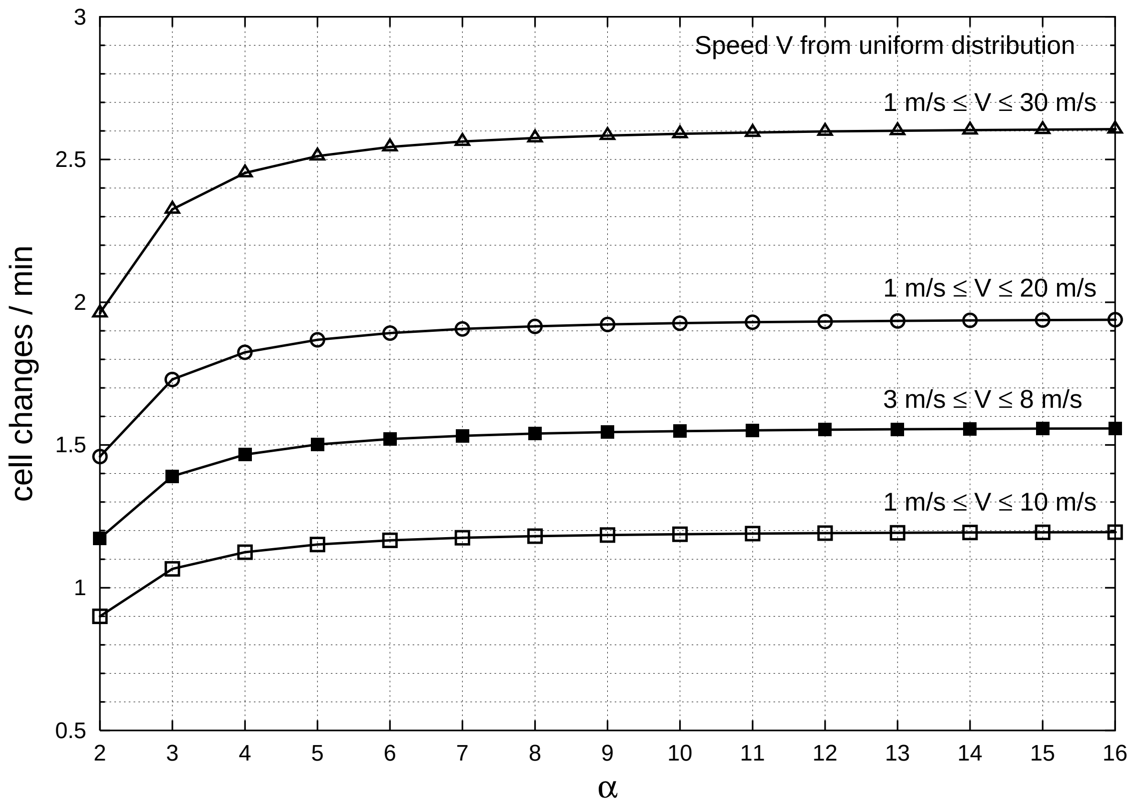


Figure 11. Expected cell change rate $E\{C_t\}$ on a square area of size $\|\mathcal{A}\| = 62500\,\alpha^2\,\mathrm{m}^2$. The number of cells is α^2. The length of one square cell is always 250 m.

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